PHYSICAL REVIEW E **72**, 015601(R) (2005)

# Dimensional effects in dynamic fragmentation of brittle materials

R. P. Linna,[1,2] J. A. Åström,[3] and J. Timonen[1]
[1]*Department of Physics, University of Jyväskylä, P. O. Box 35, FIN-40351 Jyväskylä, Finland*
[2]*Rudolf Peierls Centre for Theoretical Physics, 1 Keble Road, Oxford, OX1 3NP, United Kingdom*
[3]*CSC-Scientific Computing Ltd, P. O. Box 405, FIN-02101 Espoo, Finland*
(Received 14 March 2005; published 5 July 2005)

It has been shown previously that dynamic fragmentation of brittle $D$-dimensional objects in a $D$-dimensional space gives rise to a power-law contribution to the fragment-size distribution with a universal scaling exponent $2-1/D$. We demonstrate that in fragmentation of two-dimensional brittle objects in three-dimensional space, an additional fragmentation mechanism appears, which causes scale-invariant secondary breaking of existing fragments. Due to this mechanism, the power law in the fragment-size distribution has now a scaling exponent of $\sim 1.17$.

DOI: 10.1103/PhysRevE.72.015601 PACS number(s): 46.50.+a, 62.20.Mk

Fragmentation of brittle solids has attracted increasing interest especially after the observations of fragment-size distributions (FSD) of a power-law form [3]. Abundant experimental data are now available for objects of varying material and shape [1–6]. Several analytical and numerical models for brittle fragmentation have also been introduced [7–10]. Most of the analytical models are of hierarchical nature, i.e., a large fragment first breaks into a number of smaller fragments, which may then break further. Typically these models lead to a power-law FSD with a material dependent scaling exponent [8,9]. Very recently we have, however, produced [5,6] a model in which the power-law contribution to FSD results from mergers of side branches of the main cracks.

In this branching-merging process the smallest fragments are created first, and the process continues with increasing fragment sizes. With only few plausible assumptions, this process leads to a universal scaling exponent $2-1/D$ in the power law, which was found [5] to be in excellent agreement with the results of large-scale rock-fragmentation experiments with data extending over 12 orders of magnitude in fragment size. Recent experiments on glass tubes [11] produced a power-law contribution to FSD with this universal exponent for $D=2$. Agreement was not always equally good [6] for fragmentation of gypsum discs that had somewhat $(D=2)$-like character. Very clearly, this effect was in the early results of Refs. [1,2], according to which the exponent is smaller than the proposed universal $D=2$ value for effectively two-dimensional objects, while being fairly close to the universal $D=3$ value for really three-dimensional objects. A very clear dimensional crossover was observed, especially in Ref. [2].

Very recent experiments on the fragmentation of egg shells appeared to produce [12,13] an exponent between the $(D=2)$-like value of Refs. [1,2] and the universal $2-1/D$ for $D=2$. All these results indicate, however, that there is an additional mechanism in the fragmentation of effectively $D$-dimensional objects in a $(D+1)$-dimensional space, not captured by the branching-merging process that seems to hold for $D$-dimensional objects in a $D$-dimensional space. The experimental value of the new scaling exponents seems to be somewhat unclear, however.

In this paper we determine, by analyzing in detail the time evolution of the fragmentation of a brittle two-dimensional (2D) system embedded in a three-dimensional (3D) space, the nature of the process that leads in this case to a FSD of power-law form. We can thereby explain the difference in the fragmentation of 2D objects in a 2D space, which leads [6] to the universal exponent quoted above, and in a 3D space, which leads to a lower scaling exponent as observed in Refs. [1,2]. We can also provide a rather accurate value for this exponent. More specifically, we show that when a 2D object breaks in a 3D space, there is, in addition to the branching-merging process, a hierarchical process by which fragments already formed are further broken into smaller fragments. This secondary process also produces a power-law FSD. It is thus scale invariant, and hence the secondary cracks must be spatially correlated.

We use a numerical model in which periodic boundary conditions are imposed on a moderately curved 2D disordered brittle material (a torus) [14]. Elastic loading of this system is performed by giving an initial impulse in the outward (expansive) direction ($z$) perpendicular to the curved surface of the torus, in analogy with a "slow" explosion process. The subsequent fragmentation results from the breaking (removal) of massless beams connecting mass points, whose deformation exceeds a given breaking threshold. Motions of the mass points in the directions tangential and perpendicular to the surface are governed by discretized Newton's equations of motion. The stiffness of the surface against out-of-plane motion is regulated by making the masses at the lattice sites anisotropic so that they can be formally bigger, e.g., in the perpendicular direction, than in the tangential directions. The inertia for radial deformations will thus be increased, and the local strains in the surface will approach those in a 2D space.

First we give a short account of fragmentation by the branching-merging scheme, which will be shown below to dominate the initial fragmenting of our system. The model [5,6] was inspired by the observation [15] that small fragments were formed close to the first-formed *main cracks* whose mergers produced the large-size contribution to FSD. As the systems considered did not initially contain any major flaws from which cracks could have easily been nucleated, the elastic energy loaded in the system at the time cracks began to nucleate was relatively high. These main cracks thus propagated very fast and were unstable against formation of side branches [16].

The branching of fast cracks has been observed to appear with a well-defined distribution for the branch-to-branch intervals [16]. We can thus consider here a simplified model in which side branches are separated by a constant interval. Through long-range elastic deformation of the system around any existing crack, adjacent side branches are attracted by each other, and will eventually merge so that one branch is terminated at a free fracture surface left behind by its neighbor. It means that one branch propagates farther from each merging point of two side branches and the same process is repeated between the remaining branches. The fragments thus created are formed in succeeding "generations" defined by the numbers of mergers preceding them.

The linear size of the $i$th-generation fragments, assuming the same aspect ratio (taken here as unity) in each generation, is given by $d_i = h^i l_b$, where $l_b$ is the (average) distance of adjacent side branches and $h$ is the factor by which this distance is increased in each new generation. For the simplest case of a homogeneous process, $h=2$, but the argument can be generalized to the case in which $h$ is not a constant but approaches 2, asymptotically. The mass of the fragments in generation $i$ is given by $s_i = (h \times l_b)^{Di}$ for a $D$-dimensional system. Given an initial number $n_b$ of side branches around a main crack, the number of fragments in the $i$th generation is $n_i = n_b / [(hl_b)^{(D-1)i}]$. Taking the continuum limit such that there are $N(s) = n(s)ds$ fragments in mass interval $ds$, we easily find for the number density of fragments $n(s) \propto s^{-\alpha}$, with scaling exponent $\alpha = 2 - 1/D$. The main cracks, if uncorrelated, create an exponential contribution to FSD, and if the side branches have a finite penetration depth into the system, there will be a cut off in the power-law FSD. The full form of this FSD with an exponential cut off is given in Refs. [5,6]. In what follows we ignore for simplicity these exponential factors in FSD.

The described fragmentation proceeds from small to large fragments with exponentially increasing fragment size, and the current fragment size (in generation $i$) is linearly related to the current number of broken beams $n_{br}(i)$. The average fragment size including the residual (i.e., the nonfragmented part) in generation $i$, when $N_I$ fragments are assumed to be produced in the first phase of side-branch mergers, is given by

$$\bar{s}_i = \frac{S}{1 + N_I \sum_{i'=0}^{i} h^{-i'}} \sim \frac{1}{1 - h^{-i}}, \qquad (1)$$

where $S$ is the total mass of the system.

We follow the evolution of the fragmentation process by simulating the numerical model introduced above. Instead of the actual simulation time, we found it instructive to use the number of broken beams as the measure of time. In order to describe a real disordered material with defects, we introduced in addition a number of uncorrelated defects in terms of individual removed beams. In the simulations we used a disordered square lattice consisting of 19 450 sites connected by 38 900 beams, with Poisson distributed 4000 initial defects (a concentration of about 10%). Propagating cracks nucleated at these defects as it demanded less energy than (dynamically) creating first a new defect.

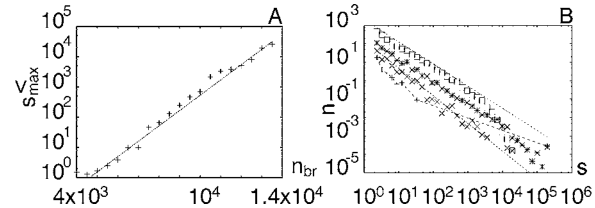


FIG. 1. (a) Average $s^<_{max}$ as a function of the number of broken beams $n_{br}$. The line fitted to the data is $\sim\exp(1.17\times10^{-3} n_{br})$. (b) The evolution of the simulated fragment-size distribution when time is measured as the number of broken beams: (+) $n_{br}=7000$, (×) $n_{br}=11\,000$, (*) $n_{br}=15\,000$, (⊡) $n_{br}=27\,000$. Lines $\sim s^{-1.5}$ and $\sim s^{-1.2}$ are also shown as guides for the eye.

Fragmentation started to appear as the system had expanded sufficiently, and the fragment-size distribution assumed initially a clear bimodal form with distinct distributions for small and large fragments. It is instructive to follow first the development of the maximum fragment size $s^<_{max}$ in the small-fragment part of the distribution. We show in Fig. 1(a) $s^<_{max}$ as a function of the number of broken beams $n_{br}$.

It is evident from this figure that $s^<_{max}$ grows exponentially with $n_{br}$, as expected for a FSD produced by the branching-merging process. The simulated FSD shown in Fig. 1(b) for a few values of $n_{br}$ confirms that, for small enough values of $n_{br}$, FSD has a power-law contribution with the universal $D=2$ exponent $\alpha=3/2$, even though the statistics of these data are not very good. (Notice that, in this logarithmic scale, the large-size part of the distribution is not clearly distinguishable, unlike in linear scale.) For increasing $n_{br}$ the exponent, however, decreases definitely below the universal value.

In order to analyze what actually happens for increasing $n_{br}$, we plot in Fig. 2(a) the averaged sizes for all fragments ($\bar{s}$) and for the largest fragment ($s_{max}$), the number of fragments in the small-size part ($N_<$), and $s^<_{max}$, all as functions of $n_{br}$. These plots clearly indicate that the branching-

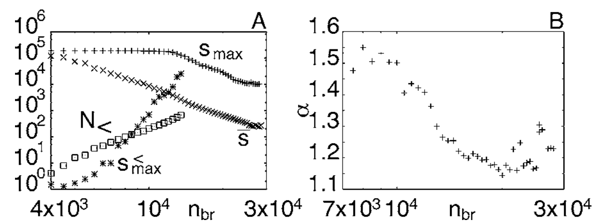


FIG. 2. Simulated evolution of the fragment-size distribution as a function of $n_{br}$. (a) (+) Average maximum fragment size $s_{max}$, (×) average fragment size $\bar{s}$, (*) average maximum fragment size of the small-size part of the distribution $s^<_{max}$, and (⊡) number of fragments in the small-size part $N_<$. (b) The scaling exponent $\alpha$ as obtained from fitting the number density of fragments by $n(s) \sim s^{-\alpha}$, as a function of $n_{br}$.

merging process continues until $s_{max}^{<}$ begins to reach the average size of all fragments. This is actually rather obvious as side branches cannot propagate farther than the average distance between the main cracks. Up to this point the maximum fragment size $s_{max}$ stays rather constant, followed by crossover to a power-law decay. This behavior of $s_{max}$ means that fragmentation does not stop when the branching-merging process stops, unlike in the previous simulations on 2D systems in a 2D space [6,10,15]. The second phase of fragmentation is dominated by fragmentation of large fragments formed during thefirst phase dominated by the branching-merging process. This is consistent with the FSD's shown in Fig. 1(b).

We show in Fig. 2(b) the scaling exponent of the simulated power-law contribution to FSD as a function of $n_{br}$. The crossover from the branching-merging process dominated first phase of fragmentation, with $\alpha \approx 1.5$, to a second phase with $\alpha \approx 1.2$, is very clearly displayed in this plot. This crossover indeed appears at the value of $n_{br}$ for which $s_{max}^{<}$ equals the average size $\bar{s}$. After the crossover the bimodal shape of FSD also disappears. We should now try to understand the nature of fragmentation in the second phase of the process after this crossover.

According to Eq. (1), for the branching-merging process $\bar{s}$ should, after an initial decrease, level off at a constant value. We can deduce from Fig. 2(a) that $\bar{s}$ decreases, however, with increasing $n_{br}$ as a power law. By inspecting in detail a series of snapshots of fragmenting systems, of which an example is given in Fig. 4, we observe, in accordance with the results shown in Fig. 1, that when the branching-merging process begins to cease, fragments already formed begin to fragment further. This secondary fragmentation process can thus be considered as hierarchical. Contrary to typical models for a hierarchical process, such as, e.g., the one in Ref. [9], the size of the largest fragment in the system does not remain constant but decreases [cf. $s_{max}$ in Fig. 2(a)]. Apparently there are no stable fragments in the secondary process. The size ratio of the new fragments produced by division of an existing fragment appears to be a constant on the average. Assuming thus that $\bar{s}_{i+1}/\bar{s}_i \approx \text{constant}$, we find $\bar{s}_i \sim \exp(-\text{const} \times i)$. For a hierarchical process with a power-law FSD, one must have $n_{br} \sim \exp(\text{const} \times i)$, and we can conclude that

$$\bar{s} \sim (n_{br})^{-\mu} \tag{2}$$

with a constant $\mu > 0$. This behavior is confirmed by simulations, cf. Fig. 2(a).

The relevant time scales related to the branching-merging and hierarchical processes are very different. The branching-merging process is initially very fast in producing the small-size part of FSD, so that the largest fragment of this part grows, as explained above, exponentially with $n_{br}$. The hierarchical process takes off very slowly, but eventually begins to form new fragments at a rate that grows exponentially with $n_{br}$, while not appreciably affecting the growth of $s_{max}^{<}$ that is still dominated by the branching-merging process. By that time the latter process produces new fragments with a rapidly decreasing growth rate. There should thus be an interval in the fragmentation process during which $s_{max}^{<} \sim N^{\beta}$

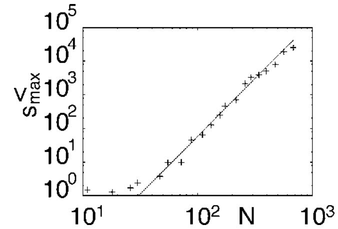


FIG. 3. $s_{max}^{<}$ as a function of the number of fragments $N$. A power-law fit gives $s_{max}^{<} \sim N^{3.44}$, which is shown as a line.

with $\beta$ a constant. Here, as before, $N$ is the total number of fragments (which depends on time). We show in Fig. 3 a simulated $s_{max}^{<}$ as a function of $N$. These data display a clear power-law dependence with $\beta \approx 3.44$.

In Fig. 4 we show a series of snapshots of the fragmentation process as described by our numerical model. The smallest fragments are indeed seen to form close to the first propagating cracks as assumed in the branching-merging process.

The FSD resulting from fragmentation seems to continue to be a power law even after the onset of the hierarchical phase of the process. In order to confirm this, we plot in Fig. 5 a differential fragment-size distribution $N_d(n_{br}) \equiv N(n_{br}) - N(n_{br}^{(o)} = 11\,500)$ for $n_{br} = 17\,000$. This differential distribution is dominated by the hierarchical process as $n_{br}^{(o)} \approx 11\,500$ is the crossover point. It is evident that the process is indeed scale invariant with a scaling exponent of $\sim 1.17$. A large-size cutoff begins to appear at $n_{br} \approx 18\,500$.

As the FSD of the hierarchical phase of the process is also scale invariant, the cracks responsible for the secondary breaking cannot be randomly distributed, which would result in an exponential FSD [6,15]. The contraction force due to

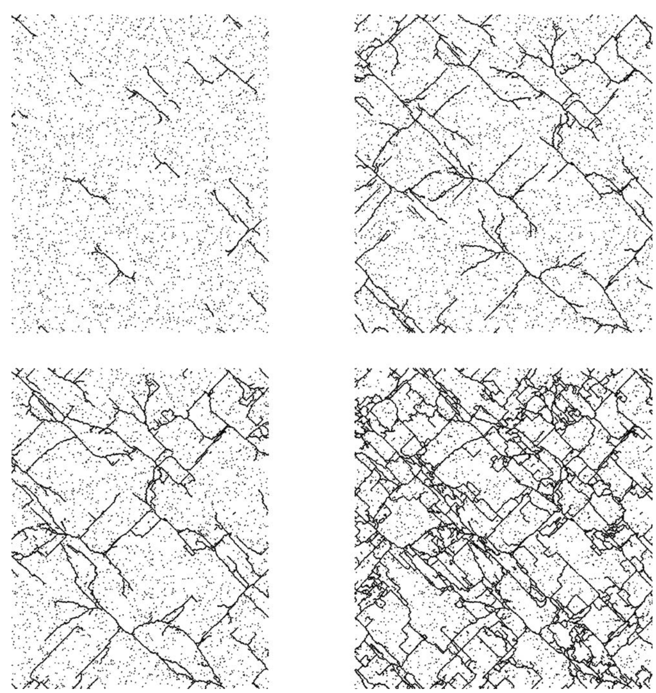

FIG. 4. Snapshots of a disordered lattice of beams undergoing fragmentation, (a) $n_{br} = 6000$, (b) $n_{br} = 12\,500$, (c) $n_{br} = 14\,500$, (d) $n_{br} = 27\,000$.

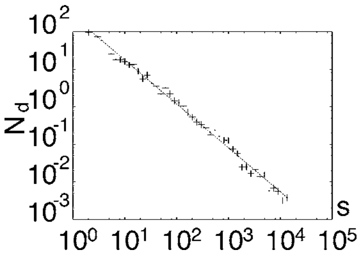


FIG. 5. Differential fragment-size distribution $N_d(n_{br}=17000)$. The power-law fit gives $N_d \sim s^{-1.17}$, shown as a line.

the curvature is now able to relax the one-dimensional crack surface through the two extra degrees of freedom per lattice site (the position and velocity in the perpendicular direction) more effectively than in the case of a surface strained in a 2D space. The expanding fragment is contracted at its edges, and the maximum strain thus appears at some distance from the crack surface. Hence, in a system of a curved surface, secondary cracks tend to be nucleated near existing crack surfaces. In this way they become spatially correlated.

In order to check that the hierarchical phase of fragmentation is indeed related to the dimension of the space being higher than that of the fragmenting object, and not, e.g., to the curvature of the surface, we varied the radius of curvature. We found identical results using, in each case, the smallest possible loading of elastic energy needed to fragment the whole system. The behavior was also found to be the same in the case when the system did not initially include broken beams.

Finally, we checked the effect of dimensionality by making the masses anisotropic such that inertia was bigger for motion in the perpendicular direction. In this way the stiffness of the out-of-plane motions is increased in comparison with the in-plane motions, and in the limit of large out-of-plane stiffness the system becomes effectively two dimensional as the additional degrees of freedom are frozen out. Simulations for increasing out-of-plane stiffness displayed a clear crossover to the universal $D=2$ scaling exponent for high values of stiffness.

In conclusion, we have shown that dynamic fragmentation of a 2D system in a 3D space begins with the nucleation of propagating cracks, and the merging of side branches formed around these cracks. This branching-merging process produces a power-law contribution to FSD just as in a 2D space, with the universal $D=2$ scaling exponent $\alpha=3/2$. Unlike in the latter case, fragmentation does not, however, stop now when the branching-merging process is terminated. It continues as a hierarchical fragmentation process that is also scale invariant. The scaling exponent related to the hierarchical FSD is $\sim 1.17$. By increasing the stiffness of the out-of-plane motions, we confirmed that there is a crossover to the universal $D=2$ scaling exponent when the third spatial dimension is effectively frozen out. Our results seem to be in excellent agreement with those of Refs. [1,2]. For fragmenting egg shells [12,13], an exponent $\sim 1.35$ was found for the power-law FSD. A possible explanation for this value is that it is related to a transient phase in which the hierarchical process was not completed (cf. Fig. 1). Also, if the full FSD with the exponential terms included were used [5,6], the data of Refs. [12] could be fitted with the universal scaling exponent $3/2$. The out-of-plane motions may well have been frozen out in fragmenting egg shells. This happens in, e.g., small gypsum fragments, which leads to a beautiful dimensional crossover in the FSD [2].

This work has been supported by the Academy of Finland (Project No. 44875). One of the authors (R.P.L.) wishes to thank Dr. Supriya Krishnamurthy for helpful discussions.

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